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ASSOCIATIVE TRIPLES AND THE YANG–BAXTER EQUATION

BY

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ABSTRACT

We introduce triples of associative algebras as a tool for building solutions to the Yang–Baxter equation. It turns out that the class of R-matrices thus obtained is related to a Hecke-like condition, which is formulated in the framework of associative algebras with non-degenerate symmetric cyclic inner product. R-matrices for a subclass of the A_n-type Belavin–Drinfel'd triples are derived in this way.

1. Introduction

The canonical Faddeev–Reshetikhin–Takhtajan recipe [RTF] for constructing quantum matrix groups is based on a solution to the Yang–Baxter equation (YBE), which is assumed to be *a priori* known. On the other hand, the theory of quantum groups, [D], was designed as an environment for constructing such solutions, which are of interest for mathematics and physics. A quantum group or a quasitriangular Hopf algebra, $\mathcal{H}$, possesses an element $\mathcal{R} \in \mathcal{H} \otimes \mathcal{H}$ called a universal R-matrix that satisfies the YBE in $\mathcal{H}^{\otimes 3}$. The universal R-matrix yields a family of matrix solutions to the YBE associated with representations of $\mathcal{H}$.

The quasi-classical limit of the YBE, the classical Yang–Baxter equation (cYBE), has a clear algebraic interpretation in terms of Manin triples, which have been classified for the semisimple Lie algebras in [BD]. The possibility to quantize an arbitrary Manin triple to a quasitriangular Hopf algebra has been proven in [EK]. In the case of semisimple Lie algebras, rather complicated although explicit formulas for universal R-matrices have been derived, [ESS]. At

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the same time, one can raise the question what structure particulars of an associative ring itself may responsible for the YBE, without appealing to the intricate technique of the Hopf algebra theory. An explicit formula for R-matrices relative to $sl_n(C)$, known as the GGS conjecture, was proposed in [GGS]. It was confirmed in many cases, [GH, Schl], and a combinatorial proof has been found recently in [Sch2]. In the present paper, we pursue another approach to the YBE, making use of the cyclic inner product in associative algebras and keeping an analogy with Manin triples. Such a point of view has led us to a definition of associative Manin (M-) triples, which allow us to explain many examples and quantize a wide class of Belavin–Drinfel'd triples associated with the special linear Lie algebras. In M-triples, R-matrices naturally split into the sum of two solutions to the YBE, one of them being a part of the canonical element of the inner product and the other belonging to a smaller subalgebra. That "smaller" solution turns out to satisfy a Hecke-like condition in algebras with non-degenerate symmetric cyclic inner product. The problem of building R-matrices in a given algebra, $\mathfrak{R}$, is thus reduced to finding an M-triple (if that is possible) with its total algebra $\mathfrak{M}$ being an extension of $\mathfrak{R}$. When $\mathfrak{R} = \mathrm{Mat}_n(\mathbb{C})$, we consider a proper extension of $\mathrm{Mat}_n(\mathbb{C}) \oplus \mathrm{Mat}_n(\mathbb{C})$ for the role of $\mathfrak{M}$.

2. Symmetric algebras and YBE

2.1. Throughout the paper, we assume $\mathfrak{M}$ to be a finite-dimensional unital associative algebra over $\mathbb{C}$. We suppose that $\mathfrak{M}$ is endowed with a symmetric cyclic inner product $(\cdot,\cdot)_{\mathfrak{M}} = (\cdot,\cdot)$, a bilinear map from $\mathfrak{M}\otimes\mathfrak{M}$ to $\mathbb{C}$ such that

$$(\alpha_1,\alpha_2) = (\alpha_2,\alpha_1), \quad (\alpha_1\alpha_2,\alpha_3) = (\alpha_2,\alpha_3\alpha_1) \quad \text{for all } \alpha_i \in \mathfrak{M}.$$

We assume $(\cdot,\cdot)$ to be non-degenerate (the induced natural map $\mathfrak{M} \to \mathfrak{M}^*$ is a linear isomorphism) and call such algebras **symmetric**. For unital algebras, symmetric cyclic inner products are in one-to-one correspondence with linear functionals $t_{\mathfrak{M}}$ obeying $t_{\mathfrak{M}}(\alpha\beta) = t_{\mathfrak{M}}(\beta\alpha) = (\alpha,\beta)_{\mathfrak{M}}$ for all $\alpha,\beta \in \mathfrak{M}$. An example is a matrix algebra with the trace functional.

Definition 1: Let $\{\alpha_i\}$ be a basis in $\mathfrak{M}$ and $\{\alpha^i\}$ its dual: $(\alpha_i,\alpha^k) = \delta_i^k$ (the Kronecker symbols). The element* $\sigma_{\mathfrak{M}} = \alpha_i\otimes\alpha^i \in \mathfrak{M}^{\otimes 2}$, which does not depend on the choice of $\{\alpha^i\}$, is called **permutation** in $\mathfrak{M}$.

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* Implicit summation over repeating upper and lower indices is understood throughout the paper.

In the endomorphism algebra of the vector space $\mathbb{C}^n$ with the trace functional, the permutation is the flip operator: $x \otimes y \mapsto y \otimes x \in \mathbb{C}^n \otimes \mathbb{C}^n$. In general, one has $(\alpha \otimes \beta)\sigma_{\mathfrak{M}} = \sigma_{\mathfrak{M}}(\beta \otimes \alpha)$ for all $\alpha, \beta \in \mathfrak{M}$.

2.2. The Yang–Baxter equation in $\mathfrak{M}^{\otimes 3}$ reads

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{1}$$

where $R \in \mathfrak{M}^{\otimes 2}$ and the subscripts specify a way of embedding $\mathfrak{M}^{\otimes 2}$ into $\mathfrak{M}^{\otimes 3}$. Assuming $R = 1 \otimes 1 + \lambda r + o(\lambda)$, $\lambda \in \mathbb{C}$, the element $r \in \mathfrak{M}^{\otimes 2}$ satisfies the classical Yang–Baxter equation in $\mathfrak{M}^{\otimes 3}$:

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0. \tag{2}$$

The square brackets mean the commutator $[\alpha_1, \alpha_2] = \alpha_1\alpha_2 - \alpha_2\alpha_1$, $\alpha_i \in \mathfrak{M}$.

Proposition 1 ([BFS]): *Permutation satisfies the YBE.*

2.3. Recall, [D], that a Manin triple $(\mathfrak{g}, \mathfrak{a}, \mathfrak{a}^*)$ comprises a Lie algebra $\mathfrak{g}$ with a non-degenerate ad-invariant symmetric inner product and its two lagrangian (maximal isotropic) Lie subalgebras $\mathfrak{a}$ and $\mathfrak{a}^*$ with the zero intersection. The canonical element $\alpha_i \otimes \beta^i \in \mathfrak{a} \otimes \mathfrak{a}^*$ solves the cYBE. To find a quantum analog of this construction, consider first the situation when a Manin triple, $(\mathfrak{M}, \mathfrak{A}, \mathfrak{A}^*)$, is formed by the commutator Lie algebras of associative algebras, and the ad-invariant two-form is at the same time a cyclic inner product in $\mathfrak{M}$. Consider $\mathfrak{A}$ and $\mathfrak{A}^*$ as bimodules over each other, the left ($\triangleright$) and right ($\triangleleft$) actions being induced via duality from the regular right and left representations. The product in $\mathfrak{M}$ is expressed by the formula

$$\alpha\beta = \alpha \triangleright \beta + \alpha \triangleleft \beta, \quad \beta\alpha = \beta \triangleright \alpha + \beta \triangleleft \alpha, \quad \alpha \in \mathfrak{A}, \quad \beta \in \mathfrak{A}^*. \tag{3}$$

Associativity is encoded in the following two equations:

$$(\alpha_1 \triangleright \beta_1, \alpha_2 \triangleleft \beta_2) + (\alpha_1 \triangleleft \beta_1, \alpha_2 \triangleright \beta_2) \\ = (\beta_1 \triangleright \alpha_2, \beta_2 \triangleleft \alpha_1) + (\beta_1 \triangleleft \alpha_2, \beta_2 \triangleright \alpha_1), \tag{4}$$

$$(\beta_2 \triangleright \alpha_2, \alpha_1 \triangleright \beta_1) + (\alpha_1 \triangleleft \beta_1, \beta_2 \triangleleft \alpha_2) = (\alpha_2\alpha_1, \beta_1\beta_2), \tag{5}$$

for all $\alpha_1, \alpha_2 \in \mathfrak{A}$ and $\beta_1, \beta_2 \in \mathfrak{A}^*$.

Conversely, given associative algebras $\mathfrak{A}$ and $\mathfrak{A}^*$ dual as linear spaces let us extend their multiplication to the linear sum $\mathfrak{A} + \mathfrak{A}^*$ by formula (3). This makes $\mathfrak{A} + \mathfrak{A}^*$ an associative algebra, $\mathfrak{A} \bowtie \mathfrak{A}^*$, provided the actions $\triangleleft$ and $\triangleright$ satisfy conditions (4) and (5). Then the natural pairing between $\mathfrak{A}$ and $\mathfrak{A}^*$ induces a non-degenerate symmetric cyclic inner product in $\mathfrak{M} = \mathfrak{A} \bowtie \mathfrak{A}^*$ such that $\mathfrak{A}$ and $\mathfrak{A}^*$ are isotropic. The following statement holds true.

PROPOSITION 2: *The canonical element $\alpha_i \otimes \alpha^i \in \mathfrak{A} \otimes \mathfrak{A}^* \subset \mathfrak{M}^{\otimes 2}$ satisfies the YBE.*

Proof: This is a corollary of Proposition 4 below. ∎

Remark 3: Let us note that $\mathfrak{A}$ from the triple $(\mathfrak{M}, \mathfrak{A}, \mathfrak{A}^*)$ is an infinitesimal bialgebra in the sense of [JR], i.e., equipped with a coassociative comultiplication $\Delta: \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$ satisfying the cocycle condition

$$\Delta(\alpha_1\alpha_2) = (1 \otimes \alpha_1)\Delta(\alpha_2) + \Delta(\alpha_1)(\alpha_2 \otimes 1) \tag{6}$$

for $\alpha_1, \alpha_2 \in \mathfrak{A}$. The map Δ is dual conjugate to the multiplication in $\mathfrak{A}^*$. It is easy to see that equation (6) is equivalent to condition (5). A coproduct is called a coboundary if $\Delta(\alpha) = (1 \otimes \alpha)r - r(\alpha \otimes 1)$, where r is an element from $\mathfrak{A} \otimes \mathfrak{A}$. For Δ to be coassociative, r is required to satisfy the *associative* Yang–Baxter equation, [A1],

$$r_{12}r_{13} - r_{23}r_{12} + r_{13}r_{23} = 0. \tag{7}$$

One can show that $\mathfrak{A} \bowtie \mathfrak{A}^*$ is a coboundary infinitesimal bialgebra with $r = \alpha_i \otimes \alpha^i$. It may be called a **double** of $\mathfrak{A}$, by analogy with the Lie bialgebra double.

Example 1: Due to condition (5), the algebras $\mathfrak{A}$ and $\mathfrak{A}^*$ cannot be simultaneously unital or they would consist of the zero elements only. Nevertheless, the algebra $\mathfrak{A} \bowtie \mathfrak{A}^*$ may have unit. Then the sum

$$1 \otimes 1 + \lambda\alpha_i \otimes \alpha^i \tag{8}$$

is also a solution to the YBE for an arbitrary value of the scalar parameter λ (cf. concluding remarks). This is true because $(\mathfrak{M}, \mathfrak{A}, \mathfrak{A}^*)$ is a Manin triple of the corresponding commutator Lie algebras. Let us give an example, rather infinite-dimensional, describing the XXX-spin chains in the theory of integrable models [F,KS]. It is related to the Yang matrix

$$R(z,u) = 1 \otimes 1 + \lambda\frac{\mathcal{P}}{z-u}, \tag{9}$$

where $\mathcal{P}$ is the conventional permutation operator acting on $\mathbb{C}^n \otimes \mathbb{C}^n$. The element $\frac{1}{z-u}$ is a brief way of writing the formal power series $\sum_{k\geq 0} \frac{u^k}{z^{k+1}}$. It represents the canonical element of pairing Res_0 between $\mathbb{C}[z]$ and $\frac{1}{z}\mathbb{C}[\frac{1}{z}]$. In this example, the algebra $\mathfrak{A} \bowtie \mathfrak{A}^*$ is formed by the Laurent polynomials with matrix coefficients, but $R(z,u)$ requires extension by the Laurent series (and an appropriate completion of tensor products).

2.4. Propositions 1 and 2 can be understood with the help of the following construction. Let $\widetilde{\mathfrak{N}}_\pm$ be two linear subspaces in $\mathfrak{M}$ mutually dual via the inner product. There exist two bijections $\widetilde{\mathfrak{N}}_\pm \to \widetilde{\mathfrak{N}}^*_\mp$ giving rise to the projectors

$$\pi_+\colon \mathfrak{M} \to \widetilde{\mathfrak{N}}_+, \quad \pi_+\colon \mu \mapsto \alpha_i(\mu, \beta^i), \tag{10}$$

$$\pi_-\colon \mathfrak{M} \to \widetilde{\mathfrak{N}}_-, \quad \pi_-\colon \mu \mapsto (\alpha_i, \mu)\beta^i, \tag{11}$$

where $\{\alpha_i\}$ is a base in $\widetilde{\mathfrak{N}}_+$ and $\{\beta^i\}$ its dual in $\widetilde{\mathfrak{N}}_-$. The choice of $\{\alpha_i\}$ does not affect $\pi_\pm$. Consider the subspaces

$$\widetilde{\mathfrak{M}}_\pm = \{\mu \in \mathfrak{M} \mid \pi_\pm(\alpha_1\mu)\alpha_2 = \alpha_1\pi_\pm(\mu\alpha_2), \forall \alpha_1, \alpha_2 \in \widetilde{\mathfrak{N}}_\pm\}. \tag{12}$$

They contain the normalizers for $\widetilde{\mathfrak{N}}_\pm$, i.e., the maximal subalgebras in $\mathfrak{M}$ for which $\widetilde{\mathfrak{N}}_\pm$ are bimodules. In particular, $\widetilde{\mathfrak{N}}_\pm \subset \widetilde{\mathfrak{M}}_\pm$ if $\widetilde{\mathfrak{N}}_\pm$ are subalgebras.

PROPOSITION 4: *Suppose $\mathfrak{M}$ is spanned by $\widetilde{\mathfrak{M}}_-$ and $\widetilde{\mathfrak{M}}_+$ as a linear space. Then the canonical element $Q = \alpha_i \otimes \beta^i \in \widetilde{\mathfrak{N}}_+ \otimes \widetilde{\mathfrak{N}}_- \subset \mathfrak{M}^{\otimes 2}$ satisfies the YBE.*

Proof: Let us check the YBE by pairing its middle tensor component with elements of $\widetilde{\mathfrak{M}}_+$ and $\widetilde{\mathfrak{M}}_-$ separately. That will be sufficient for the proof, since they span the entire $\mathfrak{M}$. For arbitrary $\mu \in \widetilde{\mathfrak{M}}_+$, the Yang–Baxter equation

$$\alpha_i\alpha_j \otimes (\beta^i\alpha_k, \mu) \otimes \beta^j\beta^k = \alpha_j\alpha_i \otimes (\alpha_k\beta^i, \mu) \otimes \beta^k\beta^j$$

can be rewritten as

$$\alpha_i\alpha_j \otimes (\beta^i, \alpha_k\mu) \otimes \beta^j\beta^k = \alpha_j\alpha_i \otimes (\beta^i, \mu\alpha_k) \otimes \beta^k\beta^j,$$

and therefore as

$$\pi_+(\alpha_k\mu)\alpha_j \otimes \beta^j\beta^k = \alpha_j\pi_+(\mu\alpha_k) \otimes \beta^k\beta^j.$$

This equation holds by the definition of $\widetilde{\mathfrak{M}}_+$, cf. formula (12). Similarly, one can check the YBE by pairing the middle tensor component with an element from $\widetilde{\mathfrak{M}}_-$. ∎

Proposition 4 is an apparent generalization of Proposition 1, because then $\widetilde{\mathfrak{N}}_\pm = \widetilde{\mathfrak{M}}_\pm = \mathfrak{M}$. It also implies Proposition 2: then one has

$$\widetilde{\mathfrak{M}}_- + \widetilde{\mathfrak{M}}_+ \supset \widetilde{\mathfrak{N}}_- \bowtie \widetilde{\mathfrak{N}}_+ = \mathfrak{M}.$$

In either case the sum of normalizers of $\widetilde{\mathfrak{N}}_\pm$ gives the whole $\mathfrak{M}$. Let us present an example where the normalizers are quite small whereas $\widetilde{\mathfrak{M}}_\pm$ coincide with $\mathfrak{M}$.

Example 2: Take $\mathfrak{M}$ to be the algebra $\mathrm{Mat}_n(\mathbb{C})$ of $n \times n$ complex matrices and denote by e^i_k the standard matrix base elements. Let σ be a permutation of the set of indices $\mathrm{I} = \{1, \ldots, n\}$. Put $\widetilde{\mathfrak{N}}_+ = \mathrm{span}\{e^{\sigma(i)}_i\}_{i\in\mathrm{I}}$ and $\widetilde{\mathfrak{N}}_- = \mathrm{span}\{e^i_{\sigma(i)}\}_{i\in\mathrm{I}}$; then the canonical element with respect to the trace pairing is

$$Q = \sum_{i\in\mathrm{I}} e^{\sigma(i)}_i \otimes e^i_{\sigma(i)}.$$

Now observe that $\widetilde{\mathfrak{M}}_+ = \mathfrak{M}$. Indeed, for any matrix $u = u^k_i e^i_k \in \mathfrak{M}$ one has $\pi_+(u e^{\sigma(i)}_i) = u^{\sigma(i)}_{\sigma(i)} e^{\sigma(i)}_i$ and $\pi_+(e^{\sigma(i)}_i u) = u^i_i e^{\sigma(i)}_i$. So one gets the identity

$$\pi_+(e^{\sigma(i)}_i u) e^{\sigma(k)}_k = u^i_i \delta^{\sigma(k)}_i e^{\sigma^2(k)}_k = u^{\sigma(k)}_{\sigma(k)} \delta^{\sigma(k)}_i e^{\sigma^2(k)}_k = e^{\sigma(i)}_i \pi_+(u e^{\sigma(k)}_k).$$

Note that the normalizers of $\widetilde{\mathfrak{N}}^\pm$ consist of diagonal matrices only.

Proposition 4 supplies us with solutions to the YBE that may seem to be quite distant from those related to quantum groups. We will use it for constructing R-matrices of real interest.

3. Associative Manin triples

3.1. Let a symmetric algebra $\mathfrak{M}$ be a linear sum (not necessarily direct) of its two subalgebras, $\mathfrak{M} = \mathfrak{M}_- + \mathfrak{M}_+$. Let $\mathfrak{N}_\pm \subset \mathfrak{M}_\pm$ denote the kernels of the inner product restricted to $\mathfrak{M}_\pm$. They are two-sided ideals in $\mathfrak{M}_\pm$.

Definition 2: We call $(\mathfrak{M}, \mathfrak{M}_+ \mathfrak{M}_-)$ an **associative Manin triple** (or simply **M-triple**) with **diagonal** $\mathfrak{D} = \mathfrak{M}_+ \cap \mathfrak{M}_-$ if the composition maps

$$\mathfrak{D} \to \mathfrak{M}_+ \to \mathfrak{M}_+/\mathfrak{N}_+, \quad \mathfrak{D} \to \mathfrak{M}_- \to \mathfrak{M}_-/\mathfrak{N}_- \tag{13}$$

are isomorphisms. A triple is called **trivial** if it coincides with $\mathfrak{D}$ and **disjoint** if $\mathfrak{D} = \{0\}$.

PROPOSITION 5: *Let $(\mathfrak{M}, \mathfrak{M}_-, \mathfrak{M}_+)$ be an M-triple. Then*
(1) *$\mathfrak{N}_-$ is dual to $\mathfrak{N}_+$ with respect to the inner product,*
(2) *$\mathfrak{N}_\pm$ are $\mathfrak{D}$-invariant,*
(3) *$\mathfrak{D}$ is orthogonal to $\mathfrak{N}_- + \mathfrak{N}_+$.*

Conversely, let $\mathfrak{M}$ be decomposed into the linear sum $\mathfrak{M} = \mathfrak{N}_- + \mathfrak{D} + \mathfrak{N}_+$ of subalgebras fulfilling these three conditions. If $\mathfrak{M}$ is unital and $1 \in \mathfrak{D}$, then $\mathfrak{M}$ and $\mathfrak{M}_\pm = \mathfrak{D} + \mathfrak{N}_\pm$ form an M-triple.

Proof: Conditions (2) and (3) readily follow from Definition 2. To prove condition (1), observe that $\mathfrak{D} \cap \mathfrak{N}_\pm = \{0\}$. This is a consequence of the isomorphisms between $\mathfrak{D}$ and $\mathfrak{M}_\pm/\mathfrak{N}_\pm$, cf. formula (13). By the same argument, restriction of $(.,.)$ to $\mathfrak{D}$ is non-degenerate. Combining this with condition (3) we find that $\mathfrak{N}_\pm$ are dual to each other, being isotropic subalgebras with zero intersection.

Conversely, suppose the algebras $\mathfrak{D}$ and $\mathfrak{N}_\pm$ in the linear decomposition of $\mathfrak{M}$ satisfy the conditions of the proposition. Condition (3) implies that restriction of $(.,.)$ to $\mathfrak{D}$ is non-degenerate. Since $\mathfrak{D}$ contains unit, the subalgebras $\mathfrak{N}_\pm$ are isotropic. Indeed, for all $\alpha_1, \alpha_2 \in \mathfrak{N}_\pm$ one has $(\alpha_1, \alpha_2) = (\alpha_1\alpha_2, 1) = 0$. Put $\mathfrak{M}_\pm = \mathfrak{D} + \mathfrak{N}_\pm$; then $\mathfrak{N}_\pm$ are the kernels of $(.,.)$ restricted to $\mathfrak{M}_\pm$ and $\mathfrak{D} = \mathfrak{M}_\pm/\mathfrak{N}_\pm$. ∎

3.2. Associative Manin triples form a category, $\mathcal{AT}$, with a subcategory $\mathcal{AT}_0$ of trivial triples. Morphisms in $\mathcal{AT}$ are algebra maps preserving the elements of triples. The category $\mathcal{AT}$ admits the following operations with objects:

(1) Transposition $\mathfrak{M}'$.

$$(\mathfrak{M}', \mathfrak{M}'_-, \mathfrak{M}'_+) = (\mathfrak{M}, \mathfrak{M}_+, \mathfrak{M}_-), \quad t_{\mathfrak{M}'} = t_{\mathfrak{M}}.$$

(2) Direct sum $\mathfrak{M}^1 \oplus \mathfrak{M}^2$.

$$(\mathfrak{M}^1 \oplus \mathfrak{M}^2, \mathfrak{M}^1_- \oplus \mathfrak{M}^2_-, \mathfrak{M}^1_+ \oplus \mathfrak{M}^2_+), \quad t_{\mathfrak{M}^1 \oplus \mathfrak{M}^2} = t_{\mathfrak{M}^1} \oplus t_{\mathfrak{M}^2}.$$

(3) Tensor product by objects of $\mathcal{AT}_0$.

$$(\mathfrak{A} \otimes \mathfrak{M}, \mathfrak{A} \otimes \mathfrak{M}_-, \mathfrak{A} \otimes \mathfrak{M}_+), \quad \mathfrak{A} \in \mathcal{AT}_0, \quad t_{\mathfrak{A} \otimes \mathfrak{M}} = t_{\mathfrak{A}} \otimes t_{\mathfrak{M}}.$$

(4) Double $D(\mathfrak{M})$.

$$D(\mathfrak{M}) = \mathfrak{D} \oplus \mathfrak{M} \oplus \mathfrak{M}, \quad D(\mathfrak{M})_+ = \mathfrak{D} \oplus \mathfrak{M} \oplus \mathrm{Id}(\mathfrak{M}),$$

$$D(\mathfrak{M})_- = \mathfrak{D} \oplus \mathfrak{D} \oplus \mathrm{Id}(\mathfrak{D}) + \{0\} \oplus \mathfrak{N}_+ \oplus \mathfrak{N}_-,$$

$$t_{D(\mathfrak{M})} = t_{\mathfrak{D}} \oplus t_{\mathfrak{M}} \ominus t_{\mathfrak{M}}.$$

In the definition of $D(\mathfrak{M})_-$ we identify the first and last copies of $\mathfrak{D}$. The diagonal of this triple is $\mathfrak{D} \oplus \mathrm{Id}(\mathfrak{D}) \oplus \mathrm{Id}(\mathfrak{D})$. In the definition of $t_{D(\mathfrak{M})}$, we assume $t_{\mathfrak{D}} = t_{\mathfrak{M}}|_{\mathfrak{D}}$, and the restriction of $t_{D(\mathfrak{M})}$ to the third addend coincides with $-t_{\mathfrak{M}}$ (that is reflected by the notation).

(5) Skew double $S(\mathfrak{M})$. This is a disjoint triple $\mathfrak{M} \bowtie \mathfrak{M}^*$ of an algebra $\mathfrak{M}$ and its dual $\mathfrak{M}^*$ equipped with the nil multiplication.

4. Associative Manin triples and YBE

4.1. In this section we use M-triples introduced in Section 3 to construct solutions to the YBE. Let us give one more definition before formulating the basic statement of the paper.

Definition 3: An element $S \in \mathfrak{M}^{\otimes 2}$, where $\mathfrak{M}$ is a symmetric algebra, is said to satisfy the **Hecke condition** in $\mathfrak{M}^{\otimes 2}$ if

$$S_{21}S - \sigma_{\mathfrak{M}}S = \lambda^2(1 \otimes 1) \tag{14}$$

for some scalar λ.

It is convenient to put $1/\lambda = \omega = q - q^{-1}$ assuming $q^2 \neq 1$ and $\lambda \neq 0$. For the matrix algebra $\mathfrak{M} = \mathrm{Mat}_n(\mathbb{C})$ with the trace pairing, this is the conventional Hecke condition. Then $\sigma^2_{\mathfrak{M}} = 1 \otimes 1$ and one can combine S with $\sigma_{\mathfrak{M}}$ getting a close quadratic equation on $\sigma_{\mathfrak{M}}S$. In general, the permutation $\sigma_{\mathfrak{M}}$ is not invertible, [BFS].

THEOREM 6: *Let $(\mathfrak{M}, \mathfrak{M}_-, \mathfrak{M}_+)$ be an M-triple with the diagonal $\mathfrak{D}$. Then the canonical element $Q \in \mathfrak{N}_+ \otimes \mathfrak{N}_-$ with respect to the inner product satisfies the YBE. Let $S \in \mathfrak{D}^{\otimes 2}$ satisfy the YBE and Hecke condition. Then the sum $R = S + Q \in \mathfrak{M}^{\otimes 2}$ is a solution to the YBE.*

Proof: The first assertion follows from Proposition 4, where we put $\widetilde{\mathfrak{N}}_\pm = \mathfrak{N}_\pm$. Indeed, $\mathfrak{N}_\pm$ are bimodules for $\mathfrak{M}_\pm \subset \widetilde{\mathfrak{M}}_\pm$ and $\mathfrak{M}_- + \mathfrak{M}_+ = \mathfrak{M}$.

Let us prove the second assertion. The element Q interacts with elements of $\mathfrak{D}$ like the permutation: $(\delta_1 \otimes \delta_2)Q = Q(\delta_2 \otimes \delta_1)$ for all $\delta_1, \delta_2 \in \mathfrak{D}$. Therefore S and Q satisfy the system of equations

$$S_{12}S_{13}Q_{23} = Q_{23}S_{13}S_{12}, \quad Q_{12}S_{13}S_{23} = S_{23}S_{13}Q_{12}. \tag{15}$$

The Yang–Baxter equation (1) for the sum $R = S + Q$ reduces to the equation

$$\begin{aligned} &S_{12}Q_{13}Q_{23} + Q_{12}S_{13}Q_{23} + Q_{12}Q_{13}S_{23} + S_{12}Q_{13}S_{23} \\ =&Q_{23}Q_{13}S_{12} + Q_{23}S_{13}Q_{12} + S_{23}Q_{13}Q_{12} + S_{23}Q_{13}S_{12}, \end{aligned} \tag{16}$$

if S is a solution to the YBE. Put $S = \delta_i \otimes \tilde{\delta}^i \in \mathfrak{D}^{\otimes 2}$ and rewrite (16) as

$$\begin{aligned} &\delta_i\alpha_j \otimes \tilde{\delta}^i\alpha_k \otimes \beta^j\beta^k + \alpha_i\delta_j \otimes \beta^i\alpha_k \otimes \tilde{\delta}^j\beta^k + \alpha_i\alpha_j \otimes \beta^i\delta_k \otimes \beta^j\tilde{\delta}^k \\ &\qquad + \delta_i\alpha_j \otimes \tilde{\delta}^i\delta_k \otimes \beta^j\tilde{\delta}^k \\ &= \alpha_j\delta_i \otimes \alpha_k\tilde{\delta}^i \otimes \beta^k\beta^j + \delta_j\alpha_i \otimes \alpha_k\beta^i \otimes \beta^k\tilde{\delta}^j + \alpha_j\alpha_i \otimes \delta_k\beta^i \otimes \tilde{\delta}^k\beta^j \\ &\qquad + \alpha_j\delta_i \otimes \delta_k\tilde{\delta}^i \otimes \tilde{\delta}^k\beta^j. \end{aligned} \tag{17}$$

It suffices to check this identity by separately pairing its middle tensor component with elements from $\mathfrak{N}_\pm$ and $\mathfrak{D}$ since they altogether span $\mathfrak{M}$.

STEP 1 (Pairing with $\alpha \in \mathfrak{N}_+$):

$$0 + \alpha_k\alpha\delta_j \otimes \tilde{\delta}^j\beta^k + \delta_k\alpha\alpha_j \otimes \beta^j\tilde{\delta}^k + 0 = 0 + \delta_j\alpha\alpha_k \otimes \beta^k\tilde{\delta}^j + \alpha_j\alpha\delta_k \otimes \tilde{\delta}^k\beta^j + 0.$$

STEP 2 (Pairing with $\beta \in \mathfrak{N}_-$):

$$\delta_i\alpha_j \otimes \beta^j\beta\tilde{\delta}^i + \alpha_i\delta_j \otimes \beta^i\beta\tilde{\delta}^j + 0 + 0 = \alpha_j\delta_i \otimes \tilde{\delta}^i\beta\beta^j + \delta_j\alpha_i \otimes \beta^i\beta\tilde{\delta}^j + 0 + 0.$$

STEP 3 (Pairing with $\delta \in \mathfrak{D}$): The first and third terms on each side turn zero. In the fourth terms, we perform the substitution $S \to \sigma_{\mathfrak{D}}$ of the last factors, employing the Hecke condition. For example,

$$S_{12}Q_{13}S_{23} = S_{12}S_{21}Q_{13} = (S_{12}(\sigma_{\mathfrak{D}})_{12} + \lambda^2)Q_{13} = S_{12}Q_{13}(\sigma_{\mathfrak{D}})_{23} + \lambda^2 Q_{13}.$$

The last term will appear on both sides of the equation and thus vanishes. The resulting equation is

$$0 + \alpha_i\delta_j \otimes \tilde{\delta}^j\delta\beta^i + 0 + \delta_i\alpha_j \otimes \beta^j\delta\tilde{\delta}^i = 0 + \delta_j\alpha_i \otimes \beta^i\delta\tilde{\delta}^j + 0 + \alpha_j\delta_i \otimes \tilde{\delta}^i\delta\beta^j,$$

and it holds identically. ∎

4.2. Introduction of associative Manin triples is motivated by the idea of reducing the YBE in $\mathfrak{M}$ to that in a smaller algebra $\mathfrak{D} \subset \mathfrak{M}$. Propositions 1 and 2 describe two extreme cases of trivial and disjoint triples providing quite exotic examples. To find more interesting solutions, one has to admit non-trivial $\mathfrak{D}$ and $\mathfrak{N}_\pm$ simultaneously. Such applications are considered in the remainder of the paper, and this section is completed with the following statement.

PROPOSITION 7: *Let $\mathfrak{M}$ be an M-triple with the diagonal $\mathfrak{D}$ and S satisfy the Hecke condition in $\mathfrak{D}^{\otimes 2}$. Then, the element $R = S + Q$ satisfies the Hecke condition in $\mathfrak{M}^{\otimes 2}$ if and only if $\sigma_{\mathfrak{D}}Q + Q^2 = 0$.*

Proof: Taking into account $S_{21}Q = QS$ and $\sigma_{\mathfrak{M}} = \sigma_{\mathfrak{D}} + Q + Q_{21}$, one has

$$\begin{aligned} R_{21}R &= S_{21}S + S_{21}Q + Q_{21}S + Q_{21}Q \\ &= \lambda(1 \otimes 1) + \sigma_{\mathfrak{M}}S + Q_{21}Q = \lambda(1 \otimes 1) + \sigma_{\mathfrak{M}}R - \sigma_{\mathfrak{D}}Q - Q^2 \end{aligned}$$

as required. ∎

Example 3 (Jimbo R-matrix for $sl_n(\mathbb{C})$, [J]): One can build, by recursion,

$$R_n = q\sum_{i=1}^{n} e_i^i \otimes e_i^i + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} e_i^i \otimes e_j^j + \omega \sum_{i<j=1}^{n} e_j^i \otimes e_i^j, \quad \omega = q - q^{-1}, \tag{18}$$

if one puts

$$\mathfrak{M} = \mathrm{Mat}_n(\mathbb{C}),\ \mathfrak{M}_+ = \sum_{i,j=1}^{n-1} \mathbb{C}e_j^i + \sum_{i=1}^{n} \mathbb{C}e_n^i \quad \text{and} \quad \mathfrak{M}_- = \sum_{i,j=1}^{n-1} \mathbb{C}e_j^i + \sum_{i=1}^{n} \mathbb{C}e_i^n.$$

The functional $t_{\mathfrak{M}}$ is taken to be the matrix trace. Thus the isotropic subalgebra $\mathfrak{N}_+$ is formed by the last matrix column without the bottom entry. Its dual $\mathfrak{N}_-$ is spanned by the bottom matrix line excepting the right-most diagonal element. The subalgebra $\mathfrak{D}$ is $\mathrm{Mat}_{n-1}(\mathbb{C}) \oplus \mathbb{C}e_n^n$. The one-dimensional R-matrix $R_1 = qe_n^n \otimes e_n^n$ fulfills the Hecke condition in $\mathrm{Mat}_1(\mathbb{C})$. Suppose, by induction, that the same holds true for the matrix $R_{n-1} \in \mathrm{Mat}_{n-1}^{\otimes 2}(\mathbb{C})$. The direct sum $R_{n-1} + R_1$ satisfies the Yang–Baxter equation but not the Hecke condition since the unit matrix $1_n \otimes 1_n$ is not equal to the sum $1_{n-1} \otimes 1_{n-1} + 1_1 \otimes 1_1$. To fix the situation, we put $\omega S = R_{n-1} + R_1 + P_{n-1} \otimes P_1 + P_1 \otimes P_{n-1}$, where P_{n-1} and P_1 stand for the projectors from $\mathbb{C}^n$ to $\mathbb{C}^{n-1} \oplus \{0\}$ and $\{0\} \oplus \mathbb{C}$. Thus defined, S solves the Yang–Baxter equation and satisfies the Hecke condition. The matrix $Q = \sum_{i=1}^{n-1} e_n^i \otimes e_i^n$ fulfills the condition of Proposition 7, so $S + Q = \frac{1}{\omega} R_n$ is the Hecke matrix by induction.

Let us consider another representation of $\mathrm{Mat}_n(\mathbb{C})$ as an M-triple, taking $\mathfrak{M}_\pm$ to be the subalgebras of upper and lower triangular matrices. Then $\mathfrak{D}$ is a commutative algebra, $\mathbb{C}^n$, formed by diagonal matrices. To construct a solution to the YBE, following Theorem 6, we should satisfy the Hecke condition (14) for the symmetric algebra $\mathfrak{D} = \mathbb{C}^n$.

Proposition 8: *A Hecke matrix in $\mathbb{C}^n \otimes \mathbb{C}^n$ has the form $S = \sum_{i,j=1}^{n} a^{ik} e_i^i \otimes e_k^k$, where*

$$a^{ii} = \pm q^{\pm 1}/\omega \quad \text{and} \quad a^{ik} = b^{ik}/\omega, \quad b^{ik} b^{ki} = 1 \tag{19}$$

for $i, k = 1, \ldots, n, \quad i \neq k$.

Proof: The Hecke condition (14) on $S = a^{ik} e_i^i \otimes e_k^k$, boils down to the system of equations

$$a^{ii} a^{ii} - a^{ii} = \lambda^2, \quad a^{ik} a^{ki} = \lambda^2, \quad k \neq i. \tag{20}$$

If one puts $\lambda = 1/\omega$, then the general solution to this system is given by (19). ∎

Note that the matrices ωS and ωR tend to $1\otimes 1$ as $q \to 1$ if one takes $a^{ii} = q/\omega$ for all $i = 1, \dots, n$. Otherwise one obtains non-quasiclassical solutions, which are not relative to Lie bialgebras. The standard R-matrix (18) corresponds to the choice $b^{ik} = 1$. The general solution deviates from (18) exactly by the Reshetikhin diagonal twist, [R].

Example 4 (Baxterization procedure): The baxterization operation converts a constant matrix solution, R, of the YBE to that with a spectral parameter,

$$R(z, u) = zR - uR_{21}^{-1}, \tag{21}$$

provided R satisfies the conventional Hecke condition

$$(\mathcal{P}R)^2 = \omega(\mathcal{P}R) + 1 \otimes 1$$

with the matrix permutation $\mathcal{P}$. The parameters z and u are usually represented in the exponential form; then (21) is a trigonometric solution to the YBE. Set $\mathfrak{N}_+ = z\mathrm{Mat}_n(\mathbb{C})[z]$, $\mathfrak{N}_- = \frac{1}{z}\mathrm{Mat}_n(\mathbb{C})[\frac{1}{z}]$, and $\mathfrak{D} = \mathrm{Mat}_n(\mathbb{C})$. The inner product in $\mathfrak{M}$ is given by the formula $(Az^k, Bz^m) = \mathrm{Tr}(AB)\delta^{k,-m}$. Thus we have built an M-triple on $\mathrm{Mat}_n(\mathbb{C})[z, \frac{1}{z}]$. Now, put $S = \frac{R}{\omega}$ and $Q = \frac{u}{z-u}\mathcal{P}$. The result will be proportional to (21) because $\omega\mathcal{P} = R - R_{21}^{-1}$. As in the case of Yang matrix (9), one has to extend the Laurent polynomial algebra by the Laurent series.

5. On quantization of Belavin–Drinfel'd triples for $sl_n(\mathbb{C})$

5.1 (Belavin–Drinfel'd triples). Consider a semisimple Lie algebra $\mathfrak{g}$ with the Cartan subalgebra $\mathfrak{h}$ and a polarization $\mathfrak{g} = \mathfrak{n}_- \dot{+} \mathfrak{h} \dot{+} \mathfrak{n}_+$. Let Δ and $\Delta^\pm$ denote, respectively, the systems of all, positive, and negative roots. Recall, [BD], that non-skew-symmetric classical r-matrices associated to $\mathfrak{g}$ are in correspondence with the Belavin–Drinfel'd (BD) triples $(\Gamma_1, \Gamma_2, \tau)$ consisting of two subsets of positive simple roots Γ_i, $i = 1, 2$, and a bijection $\tau\colon \Gamma_1 \to \Gamma_2$ preserving the lengths of the roots with respect to the Killing form. The map τ is assumed to obey the nilpotency condition: for every $\alpha \in \Gamma_1$ there is a positive integer k such that $\tau^{k-1}(\alpha) \in \Gamma_1$ and $\tau^k(\alpha) \notin \Gamma_1$. It follows from [RS] (see also [BZ]) that the double Lie algebra $D(\mathfrak{g})$ is isomorphic to the direct sum $\mathfrak{g} \oplus \mathfrak{g}$ with the invariant inner product given by the difference of the Killing forms on the two addends. The geometric description of $\mathfrak{g}^*$ as a Lie subalgebra in $D(\mathfrak{g})$ is as follows, [S]. Let $\Delta_i \subset \Delta$ be the subsystems of roots generated by Γ_i, $i = 1,2$. Denote by $\mathfrak{g}_i \subset \mathfrak{g}$ the semisimple Lie algebras with the root systems Δ_i, and the Cartan subalgebras $\mathfrak{h}_i$. The bijection τ is extended to a Lie algebra isomorphism $\tau\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\mathfrak{g}_1 \oplus \tau(\mathfrak{g}_1)$ has the trivial intersection with the diagonal embedding $\mathfrak{g} \oplus \mathrm{Id}(\mathfrak{g})$. In other words, τ has no stable points. The isotropic Lie subalgebra $(\mathfrak{g}_1 + \mathfrak{n}_+) \oplus (\mathrm{Id}(\mathfrak{g}_1) + n_-)$ lies in $\mathfrak{g}^*$, and its complement (whose exact form is irrelevant for our consideration) in $\mathfrak{g}^*$ belongs to $\mathfrak{h} \oplus \mathfrak{h}$. The classical r-matrix for $\mathfrak{g}$ is recovered from the canonical element of the pairing between $\mathfrak{g} \oplus \mathrm{Id}(\mathfrak{g})$ and $\mathfrak{g}^*$ by projecting $\mathfrak{g} \oplus \mathfrak{g}$ to one of the addends.

5.2 (Associative BD triples). We restrict our further consideration to the case $\mathfrak{g} = sl_n(\mathbb{C})$. The triangular decomposition of $\mathfrak{g}$ is taken to be that into the diagonal, strictly upper- and lower-triangular matrices. The double $D(\mathfrak{g})$ is represented in the direct sum of two matrix algebras $\mathfrak{R} = \mathrm{Mat}_n(\mathbb{C})$. The cyclic inner product in $\mathfrak{R}^2$ is induced by the functional $t_{\mathfrak{R}^2} = \mathrm{Tr} \ominus \mathrm{Tr}$, the difference of the corresponding traces.

Denote by $\mathcal{A}(\mathfrak{l})$ the associative envelope of a Lie algebra $\mathfrak{l}$ in $\mathfrak{R}^2$. Clearly $\mathcal{A}(\mathfrak{g} \oplus \mathrm{Id}(\mathfrak{g})) = \mathfrak{R} \oplus \mathrm{Id}(\mathfrak{R})$, while $\mathcal{A}(\mathfrak{g}_i)$, $i = 1,2$, are block-diagonal subalgebras in $\mathfrak{R}$. Suppose that τ may be extended to the (unique) algebra isomorphism

$$\hat{\tau}\colon \mathcal{A}(\mathfrak{g}_1) \to \mathcal{A}(\mathfrak{g}_2). \tag{22}$$

That is the case when and only when τ preserves the orientation of the connected components in Γ_i that is induced by an orientation of the Dynkin diagram. Such Belavin–Drinfel'd triples are called **associative** in [Sch3].

Denote by $\mathfrak{H}$ the subalgebra of diagonal matrices in $\mathfrak{R}$. By projection to an ideal in $\mathfrak{H}$ or $\mathfrak{H}^2$ we understand the projection along the complementary ideal. Let $\hat{\Gamma} \subset \mathfrak{R}$ denote the set of diagonal rank-one matrix idempotents $\eta_i = e_i^i \in \mathfrak{H}$, $i = 1, \ldots, n$, and $\hat{\Gamma}_i = \hat{\Gamma} \cap \mathcal{A}(\mathfrak{g}_i)$, $i = 1,2$. The homomorphism $\hat{\tau}$ defines a bijection $\hat{\Gamma}_1 \to \hat{\Gamma}_2$, for which we use the same notation. We impose a condition on this bijection assuming for every $\eta \in \hat{\Gamma}_1$ there is the smallest positive integer $m(\eta)$ such that $\hat{\tau}^{m(\eta)}(\eta) \notin \hat{\Gamma}_1$. Like in BD triples, this condition means that the map (22) has no stable points. This requirement holds, for example, if $\hat{\Gamma}_1 \cap \hat{\Gamma}_2 = \emptyset$ or $\hat{\tau}(\eta_i) = \eta_k \Rightarrow k > i$.

5.3 (From associative BD to M-triples). Denote by $\mathfrak{B}$ the associative subalgebra in $(\mathcal{A}(\mathfrak{g}_1) + n_+) \oplus (\mathcal{A}(\mathfrak{g}_2) + n_-)$ obtained by identification of $\mathcal{A}(\mathfrak{g}_1)$ and $\mathcal{A}(\mathfrak{g}_2)$ via $\hat{\tau}$. We define $\mathfrak{d}_1$ and $\mathfrak{d}_2$ to be the ideals in $\mathfrak{H}^2$ spanned by $\hat{\Gamma} \backslash \hat{\Gamma}_1 \oplus \{0\}$ and $\{0\} \oplus \hat{\Gamma} \backslash \hat{\Gamma}_2$ correspondingly. Both $\mathfrak{d}_i$ have dimension $n - m$, where $m = \# \hat{\Gamma}_1$. The subspace $\mathfrak{B}$ is a two-sided module over the algebras $\mathfrak{d}_i$, so the sum $\mathfrak{B} + \mathfrak{d}_1 + \mathfrak{d}_2$ is closed under the multiplication. Its intersection $\mathfrak{d}$ with $\mathfrak{R}\oplus\mathrm{Id}(\mathfrak{R})$ is a subalgebra in $\mathfrak{H}^2$.

LEMMA 9: *Projectors* $\gamma_i\colon \mathfrak{d}\to\mathfrak{d}_i$, $i=1,2$, *have the full rank* $n-m$.

Proof: We will prove the statement only for $i=2$ in view of the symmetry $1\leftrightarrow 2$, $\hat\tau\leftrightarrow\hat\tau^{-1}$. Let $\eta\in\Gamma\backslash\Gamma_2$. Define the sets $\hat\Gamma_\eta=\{\eta,\hat\tau(\eta),\dots,\hat\tau^{m(\eta)}(\eta)\}$ if $\eta\in\hat\Gamma_1\backslash\hat\Gamma_2$ and $\hat\Gamma_\eta=\{\eta\}$ in case $\eta\in\hat\Gamma\backslash(\hat\Gamma_1\cup\hat\Gamma_2)$. They do not intersect for different η and clearly $\hat\Gamma_\eta\cap(\hat\Gamma\backslash\hat\Gamma_2)=\eta$. The one-dimensional subspace in $\mathfrak{H}\oplus\mathfrak{H}$ spanned by the element $(\sum_{\xi\in\hat\Gamma_\eta}\xi)\oplus(\sum_{\xi\in\hat\Gamma_\eta}\xi)$ is evidently contained in $\mathfrak{R}\oplus\mathrm{Id}(\mathfrak{R})$. It is also contained in $\mathfrak{B}+\mathfrak{d}_1+\mathfrak{d}_2$ because its projection to $\mathbb{C}\hat\Gamma_1\oplus\mathbb{C}\hat\Gamma_2$ lies in the subalgebra of $\mathfrak{B}$ spanned by $\hat\Gamma_1\oplus\hat\tau(\hat\Gamma_1)$. Hence it is a subspace in $\mathfrak{d}$, and its projection to $\mathfrak{d}_2$ is $\mathbb{C}\eta$. ∎

COROLLARY 10: *Algebras* $\mathfrak{d}_i$, $i=1,2$, *and* $\mathfrak{B}$ *are two-sided* $\mathfrak{d}$*-modules.*

Proof: The statement is obvious in what concerns $\mathfrak{d}_i$, because they are ideals in $\mathfrak{H}\oplus\mathfrak{H}$. The sum $\mathfrak{B}+\mathfrak{d}_1+\mathfrak{d}_2$ is a direct sum of modules over its subalgebra of diagonal matrices, which contains $\mathfrak{d}$. This proves the statement for $\mathfrak{B}$. ∎

Let us introduce the algebras

$$\begin{aligned}\mathfrak{M}&=\mathfrak{d}_2\oplus\mathfrak{R}\oplus\mathfrak{R}, & \mathfrak{N}_+&=\gamma_2(\mathfrak{d})\oplus\{0\}\oplus\gamma_2(\mathfrak{d})+\{0\}\oplus\mathfrak{B},\\ \mathfrak{D}&=\gamma_2(\mathfrak{d})\oplus\mathfrak{d}, & \mathfrak{N}_-&=\{0\}\oplus\mathfrak{R}\oplus\mathrm{Id}(\mathfrak{R}).\end{aligned}\tag{23}$$

The non-degenerate cyclic inner product in $\mathfrak{M}$ is defined via the functional $t_{\mathfrak{M}}=t_{\mathfrak{d}_2}\oplus t_{\mathfrak{R}^2}=t_{\mathfrak{d}_2}\oplus t_{\mathfrak{R}}\ominus t_{\mathfrak{R}}$, where $t_{\mathfrak{d}_2}$ is $t_{\mathfrak{R}}$ restricted to $\mathfrak{d}_2$.

THEOREM 11: *Let* (Γ_1,Γ_2,τ) *be a BD triple and* τ *extends to the isomorphism* $\hat\tau\colon \mathcal{A}(\mathfrak{g}_1)\to\mathcal{A}(\mathfrak{g}_2)$ *of associative algebras with no stable points. Then, the algebras* $\mathfrak{M}$ *and* $\mathfrak{M}_\pm=\mathfrak{D}+\mathfrak{N}_\pm$, *where* $\mathfrak{D}$ *and* $\mathfrak{N}_\pm$ *are defined by* (23), *form an M-triple.*

Proof: First of all observe that $\mathfrak{M}$ is a symmetric algebra. It is easy to deduce from Lemma 9 that $\mathfrak{N}_\pm\cap\mathfrak{D}=\mathfrak{N}_+\cap\mathfrak{N}_-=\{0\}$. Further, $\dim\mathfrak{N}_\pm=n$ and $\dim\mathfrak{D}=\dim\mathfrak{d}$, from which we conclude that $\mathfrak{M}=\mathfrak{N}_-+\mathfrak{D}+\mathfrak{N}_-$. Since $\mathfrak{D}$ is a unital subalgebra in $\mathfrak{M}$, it suffices to satisfy the conditions of Proposition 5, in order to prove the theorem. Observe that $\mathfrak{N}_\pm$ are $\mathfrak{D}$-invariant. That is obvious for $\mathfrak{N}_-$ and follows from Corollary 10 for $\mathfrak{N}_+$. It is easy to see that $\mathfrak{N}_\pm$ are isotropic. Along with $\mathfrak{D}$-invariance, this implies that $\mathfrak{N}_\pm$ are orthogonal to $\mathfrak{D}$. It remains to show that the pairing between $\mathfrak{N}_-$ and $\mathfrak{N}_+$ is non-degenerate. This is a consequence of the following two facts: firstly, the restriction of the inner product to $\mathfrak{D}$ is non-degenerate and, secondly, $\mathfrak{N}_\pm$ are isotropic and orthogonal to $\mathfrak{D}$. ∎

5.4 (THE R-MATRIX). To build the R-matrix associated with a BD triple specialized in Theorem 11, let us compute the element $Q \in \mathfrak{N}_+ \otimes \mathfrak{N}_-$ first. Let e_γ and f_γ, $\gamma \in \Delta^+$, denote respectively positive and negative root vectors normalized to $(e_\gamma, f_\gamma) = 1$ with respect to the trace pairing. We can choose them among the standard matrix base $\{e^i_j\}$. Set $\hat{\tau}_1 = \hat{\tau}$ and $\hat{\tau}_2 = \hat{\tau}^{-1}$. Assuming $i = 1, 2$, let us define the functions $m_i\colon \Delta \to \mathbb{Z}$ in the following way. For $\gamma \in \Delta_i$ we set $m_i(\gamma)$ to be the smallest positive integer such that $\hat{\tau}_i(\gamma) \notin \Delta_i$; if $\gamma \notin \Delta_i$, we put $m_i(\gamma) = 0$. We also introduce two-valued functions, θ_i, on Δ_i by setting $\theta_i(\gamma) = 1$ if $\gamma \in \Delta_i$ and $\theta_i(\gamma) = 0$ otherwise.

By construction (23) of $\mathfrak{N}_\pm$, the element Q coincides with the canonical element of the pairing between $\mathfrak{g} \oplus \mathrm{Id}(\mathfrak{g})$ and $\mathfrak{g}^*$ in the off-diagonal sector. This part of Q includes the following two addends:

$$(0 \oplus e_\gamma \oplus \theta_1(\gamma) e_{\hat{\tau}(\gamma)}) \otimes \sum_{k=0}^{m_2(\gamma)} (0 \oplus f_{\hat{\tau}^{-k}(\gamma)} \oplus f_{\hat{\tau}^{-k}(\gamma)}), \tag{24}$$

$$-(0 \oplus \theta_2(\gamma) f_{\hat{\tau}^{-1}(\gamma)} \oplus f_\gamma) \otimes \sum_{k=0}^{m_1(\gamma)} (0 \oplus e_{\hat{\tau}^k(\gamma)} \oplus e_{\hat{\tau}^k(\gamma)}). \tag{25}$$

The part of Q involving diagonal matrices comprises

$$-(\eta \oplus 0 \oplus \eta) \otimes (0 \oplus \eta \oplus \eta), \quad \eta \notin \hat{\Gamma}_2 \cup \hat{\Gamma}_1, \tag{26}$$

$$-\sum_{k=0}^{m(\eta)} (\eta \oplus 0 \oplus \eta) \otimes (0 \oplus \hat{\tau}^k(\eta) \oplus \hat{\tau}^k(\eta)), \quad \eta \in \hat{\Gamma}_1 \backslash \hat{\Gamma}_2, \tag{27}$$

$$-\sum_{k=0}^{m(\eta)} (0 \oplus \hat{\tau}^{-1}(\eta) \oplus \eta) \otimes (0 \oplus \hat{\tau}^k(\eta) \oplus \hat{\tau}^k(\eta)), \quad \eta \in \hat{\Gamma}_2 \cap \hat{\Gamma}_1, \tag{28}$$

$$-(0 \oplus \hat{\tau}^{-1}(\eta) \oplus \eta) \otimes (0 \oplus \eta \oplus \eta), \quad \eta \in \hat{\Gamma}_2 \backslash \hat{\Gamma}_1. \tag{29}$$

The element Q is the sum of terms (24)–(29).

To construct the R-matrix on the base of Theorem 11, we should satisfy the Hecke condition for some $S \in \mathfrak{D}^{\otimes 2}$. This problem is solved in Proposition 8, formula (19), since $\mathfrak{D}$ is isomorphic to $\mathbb{C}^{n-m}$ as a symmetric algebra. It follows from the proof of Lemma 9 that the isomorphism $\mathbb{C}^{n-m} \to \mathfrak{D}$ is given by the correspondence

$$\hat{\Gamma}\backslash\hat{\Gamma}_2 \ni \eta \mapsto \sum_{k,i=0}^{m(\eta)} (\eta \oplus \hat{\tau}^k(\eta) \oplus \hat{\tau}^i(\eta)).$$

Here, we assume $m(\eta) = 0$ if $\eta \notin \hat{\Gamma}_1$.

Finally, to obtain the R-matrix in $\mathfrak{R}^{\otimes 2}$, let us apply the projection π: $\mathfrak{M} \to \{0\} \oplus \mathfrak{R} \oplus \{0\}$. The result is

$$R = (\pi \otimes \pi)(S) + (\pi \otimes \pi)(Q), \tag{30}$$

where

$$(\pi \otimes \pi)(S) = \sum_{\eta_i, \eta_j \in \hat{\Gamma} - \hat{\Gamma}_2} a^{ij} \sum_{l=0}^{m(\eta_i)} \hat{\tau}^l(\eta_i) \otimes \sum_{k=0}^{m(\eta_j)} \hat{\tau}^k(\eta_j), \tag{31}$$

$$(\pi \otimes \pi)(Q) = -\sum_{\eta \in \hat{\Gamma}_1} \sum_{k=1}^{m(\eta)} \eta \otimes \hat{\tau}^k(\eta) + \sum_{\gamma \in \Delta^+} e_\gamma \otimes f_\gamma + \sum_{\sigma, \gamma \in \Delta^+,\ \sigma \prec \gamma} e_\gamma \wedge f_\sigma. \tag{32}$$

The symbol $\prec$ means the partial ordering in Δ^+ which is defined by τ: $\sigma \prec \gamma$ if $\tau^k(\sigma) = \gamma$ for some $k > 0$. The numbers a^{ij} provide the principal solution of the Hecke condition for $\mathbb{C}^{n-m}$ as stated by Proposition 8. Let us emphasize that, besides quasi-classical R-matrices, we also obtain those which do not tend to unit as $q \to 1$. For that, we take $a^{ii} = -q^{-1}/\omega$ rather than q/ω for some $i = 1, \ldots, n-m$.

Example 5: The formulas (30)–(32) give the standard Jimbo R-matrix (18) corresponding to empty $\hat{\Gamma}_i$. Indeed, the first and third terms in the expression for $(\pi\otimes\pi)(Q)$ vanish, and summation over $l > 0$ and $k > 0$ in $(\pi\otimes\pi)(S)$ is cancelled. Another extreme possibility is the BD triple with $\hat{\Gamma}_1 = \{\eta_1, \eta_2, \ldots, \eta_{n-1}\}$, $\hat{\Gamma}_2 = \{\eta_2, \eta_3, \ldots, \eta_n\}$, and the isomorphism $\hat{\tau}$: $\eta_i \mapsto \eta_{i+1}$. It leads to the solution to the YBE called the Cremmer–Gervais R-matrix, [CG]. In this case, the algebra $\mathfrak{D}$ is one-dimensional, so the Hecke condition is evidently fulfilled for any scalar $S = \lambda$. Thus we come to the one-parameter Cremmer–Gervais solution in the form $\lambda + (\pi \otimes \pi)(Q)$. This is in agreement with [H1]. Putting $\lambda = q/\omega$, we get the R-matrix of [H2] for the special value of the parameter $p = 1$.

6. Concluding remarks

6.1. Let us note that a representation of the Cremmer–Gervais R-matrix by the sum of two solutions to the YBE was used in [H1]. Such a representation reduces the YBE to a relation between those solutions (S and Q in our notation). The associative Manin triples introduced in the present paper is an algebraic construction designed to solve the relation of a special type, namely, expressed by equations (15) and (16). We gave numerous examples of such triples including those arising in the Lie bialgebra theory and the Belavin–Drinfel'd triples for $sl(n)$. We did not incorporate all the BD triples into our scheme, but only those falling into the class appearing in the theory of associative YBE, [Sch3]. We expect that our construction is applicable to all the BD triples considered in [Sch3].

6.2. We would like to demonstrate how an extension of our construction explains deformation of the Yang R-matrix with a constant unitary R-matrix, [BFS]. According to [BFS], the unit $1 \otimes 1$ in formula (9) may be replaced by any matrix $S \in \mathrm{Mat}_n(\mathbb{C}) \otimes \mathrm{Mat}_n(\mathbb{C})$ subject to the unitary condition $S_{21}S = 1 \otimes 1$. Below, we propose a construction representing the R-matrix as the sum $S + Q$ of two solutions to the YBE satisfying equations (15) and (16). However, the underlying mechanism is different from what was employed in Theorem 6.

Consider a disjoint triple $\mathfrak{M} = \mathfrak{M}_- \bowtie \mathfrak{M}_+$ and select the subspaces $\mathfrak{M}^c_\pm$ of "constants" in $\mathfrak{M}_\pm$ consisting of elements annihilated by the actions $\triangleright$ and $\triangleleft$: $\mathfrak{M}_\pm \triangleright \mathfrak{M}^c_\mp = \{0\} = \mathfrak{M}^c_\mp \triangleleft \mathfrak{M}^c_\pm$. The subspaces $\mathfrak{M}^c_\pm$ are subalgebras, as follows from (5), and the sum $\mathfrak{M}^c = \mathfrak{M}^c_- + \mathfrak{M}^c_+$ is direct. In the case $\mathfrak{M} = \mathrm{Mat}_n(\mathbb{C})[z, \frac{1}{z}]$, $\mathfrak{M}_+ = \mathrm{Mat}_n(\mathbb{C})[z]$, $\mathfrak{M}_- = \frac{1}{z}\mathrm{Mat}_n(\mathbb{C})[\frac{1}{z}]$ considered in Example 1, $\mathfrak{M}^c$ coincides with $\mathrm{Mat}_n(\mathbb{C})$. From (3) one has $(\alpha \otimes \beta)Q = Q(\beta \otimes \alpha)$ for $\alpha, \beta \in \mathfrak{M}^c$. Therefore equations (15) are satisfied if $S \in \mathfrak{M}^c \otimes \mathfrak{M}^c$. If S solves the YBE, then the YBE for $S + Q$ is equivalent to (16). Since Q solves simultaneously the classical and associative YBE (cf. formulas (2) and (7)), equation (16) holds, provided S satisfies the unitary condition $SS_{21} = 1 \otimes 1$.

6.3. An analysis shows that the developed technique cannot be applied, as it is, to the other series of simple Lie algebras. An extension of our approach to that case is an open problem.

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